# INTEGRABLE AND WEYL MODULES FOR QUANTUM AFFINE $sl_2$.

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## 0. Introduction

Let $\mathfrak{t}$ be an arbitrary symmetrizable Kac-Moody Lie algebra and $\mathbf{U}_q(\mathfrak{t})$ the corresponding quantized enveloping algebra of $\mathfrak{t}$ defined over $\mathbf{C}(q)$. If $\mu$ is a dominant integral weight of $\mathfrak{t}$ then one can associate to it in a natural way an irreducible integrable $\mathbf{U}_q(\mathfrak{t})$-module $L(\mu)$. These modules have many nice properties and are well understood, [K], [L1].

More generally, given any integral weight $\lambda$, Kashiwara [K] defined an integrable $\mathbf{U}_q(\mathfrak{t})$-module $V^{max}(\lambda)$ generated by an extremal vector $v_\lambda$. If $w$ is any element of the Weyl group $W$ of $\mathfrak{t}$, then one has $V^{max}(\lambda) \cong V^{max}(w\lambda)$. Further, if $\lambda$ is in the Tits cone, then $V^{max}(\lambda) \cong L(w_0\lambda)$, where $w_0 \in W$ is such that $w_0\lambda$ is dominant integral. In the case when $\lambda$ is not in the Tits cone, the module $V^{max}(\lambda)$ is not irreducible and very little is known about it, although it is known that it admits a crystal basis, [K].

In the case when $\mathfrak{t}$ is an affine Lie algebra, an integral weight $\lambda$ is not in the Tits cone if and only if $\lambda$ has level zero. Choose $w_0 \in W$ so that $w_0\lambda$ is dominant with respect to the underlying finite-dimensional simple Lie algebra of $\mathfrak{t}$. In as yet unpublished work, Kashiwara proves that $V^{max}(\lambda) \cong W_q(w_0\lambda)$, where $W_q(w_0\lambda)$ is an integrable $\mathbf{U}_q(\mathfrak{t})$-module defined by generators and relations analogous to the definition of $L(\mu)$.

In [CP4], we studied the modules $W_q(\lambda)$ further. In particular, we showed that they have a family $W_q(\boldsymbol{\pi})$ of non–isomorphic finite-dimensional quotients which are maximal, in the sense that any another finite-dimensional quotient is a proper quotient of some $W_q(\boldsymbol{\pi})$. In this paper, we show that, if $\mathfrak{t}$ is the affine Lie algebra associated to $sl_2$ and $\lambda = m \in \mathbf{Z}^+$, the modules $W_q(\boldsymbol{\pi})$ all have the same dimension $2^m$. This is done by showing that the modules $W_q(\boldsymbol{\pi})$, under suitable conditions, have a $q = 1$ limit, which allows us to reduce to the study of the corresponding problem in the classical case carried out in [CP4]. The modules $W_q(\boldsymbol{\pi})$ have a unique irreducible quotient $V_q(\boldsymbol{\pi})$, and we show that these are all the irreducible finite-dimensional $\mathbf{U}_q(\mathfrak{t})$-modules. In [CP1], [CP2], a similar classification was obtained by regarding $q$ as a complex number and $\mathbf{U}_q(\mathfrak{t})$ as an algebra over $\mathbf{C}$; in the present situation, we have to allow modules defined over finite extensions of $\mathbf{C}(q)$.

We are then able to realize the modules $W_q(m)$ as being the space of invariants of the action of the Hecke algebra $\mathcal{H}_m$ on the tensor product $(V \otimes \mathbf{C}(q)[t, t^{-1}])^{\otimes m}$, where $V$ is a two-dimensional vector space over $\mathbf{C}(q)$. Again, this is done by reducing to the case of $q = 1$.

In the last section, we indicate how to extend some of the results of this paper to the general case. We conjecture that the dimension of the modules $W_q(\boldsymbol{\pi})$ depends only on $\lambda$, and we give a formula for this dimension.

## 1. Preliminaries and Some Identities

Let $sl_2$ be the complex Lie algebra with basis $\{x^+, x^-, h\}$ satisfying

$$[x^+, x^-] = h, \quad [h, x^\pm] = \pm 2x^\pm.$$

Let $\mathfrak{h} = \mathbf{C}h$ be the Cartan subalgebra of $sl_2$, let $\alpha \in \mathfrak{h}^*$ the positive root of $sl_2$, given by $\alpha(h) = 2$, and set $\omega = \alpha/2$. Let $s : \mathfrak{h}^* \to \mathfrak{h}^*$ be the simple reflection given by $s(\alpha) = -\alpha$.

The extended loop algebra of $sl_2$ is the Lie algebra

$$L^e(\mathfrak{g}) = sl_2 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}d,$$

with commutator given by

$$[d, x \otimes t^r] = rx \otimes t^r, \quad [x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$$

for $x, y \in sl_2$, $r, s \in \mathbf{Z}$. The loop algebra $L(\mathfrak{g})$ is the subalgebra $sl_2 \otimes \mathbf{C}[t, t^{-1}]$ of $L^e(\mathfrak{g})$. Let $\mathfrak{h}^e = \mathfrak{h} \oplus \mathbf{C}d$. Define $\delta \in (\mathfrak{h}^e)^*$ by

$$\delta(\mathfrak{h}) = 0, \quad \delta(d) = 1.$$

Extend $\lambda \in \mathfrak{h}^*$ to an element of $(\mathfrak{h}^e)^*$ by setting $\lambda(d) = 0$. Set $P^e = \mathbf{Z}\omega \oplus \mathbf{Z}\delta$, and define $P^e_+$ in the obvious way. We regard $s$ as acting on $(\mathfrak{h}^e)^*$ by setting $s(\delta) = \delta$.

For any $x \in sl_2$, $m \in \mathbf{Z}$, we denote by $x_m$ the element $x \otimes t^m \in L^e(\mathfrak{g})$. Set

$$e_1^\pm = x^\pm \otimes 1, \quad e_0^\pm = x^\mp \otimes t^{\pm 1}.$$

Then, the elements $e_i^\pm$, $i = 0, 1$, and $d$ generate $L^e(\mathfrak{g})$.

For any Lie algebra $\mathfrak{a}$, the universal enveloping algebra of $\mathfrak{a}$ is denoted by $\mathbf{U}(\mathfrak{a})$. We set

$$\mathbf{U}(L^e(\mathfrak{g})) = \mathbf{U}^e, \quad \mathbf{U}(L(\mathfrak{g})) = \mathbf{U}, \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}^{\rm fin}.$$

Let $\mathbf{U}(<)$ (resp. $\mathbf{U}(>)$) be the subalgebra of $\mathbf{U}$ generated by the $x_m^-$ (resp. $x_m^+$) for $m \in \mathbf{Z}$. Set $\mathbf{U}^{\rm fin}(<) = \mathbf{U}(<) \cap \mathbf{U}^{\rm fin}$ and define $\mathbf{U}^{\rm fin}(>)$ similarly. Finally, let $\mathbf{U}(0)$ be the subalgebra of $\mathbf{U}$ generated by the $h_m$ for all $m \neq 0$. We have

$$\mathbf{U}^{\rm fin} = \mathbf{U}^{\rm fin}(<)\mathbf{U}(\mathfrak{h})\mathbf{U}^{\rm fin}(>),$$
$$\mathbf{U}^e = \mathbf{U}(<)\mathbf{U}(0)\mathbf{U}(\mathfrak{h}^e)\mathbf{U}(>).$$

Now let $q$ be an indeterminate, let $\mathbf{K} = \mathbf{C}(q)$ be the field of rational functions in $q$ with complex coefficients, and let $\mathbf{A} = \mathbf{C}[q, q^{-1}]$ be the subring of Laurent polynomials. For $r, m \in \mathbf{N}$, $m \geq r$, define

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = [m][m-1]\ldots[2][1], \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!}.$$

Then, $\begin{bmatrix} m \\ r \end{bmatrix} \in \mathbf{A}$ for all $m \geq r \geq 0$.

Let $\mathbf{U}_q^e$ be the quantized enveloping algebra over $\mathbf{K}$ associated to $L^e(\mathfrak{g})$. Thus, $\mathbf{U}_q^e$ is the quotient of the quantum affine algebra obtained by setting the central generator equal to 1. It follows from [Dr], [B], [J] that $\mathbf{U}_q^e$ is the algebra with generators $\mathbf{x}_r^\pm$ $(r \in \mathbf{Z})$, $K^{\pm 1}$, $\mathbf{h}_r$ $(r \in \mathbf{Z}\backslash\{0\})$, $D^{\pm 1}$, and the following defining

relations:

$$
\begin{gathered}
KK^{-1} = K^{-1}K = 1, \quad DD^{-1} = D^{-1}D = 1, \quad DK = KD, \\
K\mathbf{h}_r = \mathbf{h}_r K, \;\; K\mathbf{x}_r^{\pm} K^{-1} = q^{\pm 2}\mathbf{x}_r^{\pm}, \\
D\mathbf{x}_r^{\pm} D^{-1} = q^r \mathbf{x}_r^{\pm}, \;\; D\mathbf{h}_r D^{-1} = q^r \mathbf{h}_r, \\
[\mathbf{h}_r, \mathbf{h}_s] = 0, \;\; [\mathbf{h}_r, \mathbf{x}_s^{\pm}] = \pm\frac{1}{r}[2r]\mathbf{x}_{r+s}^{\pm}, \\
\mathbf{x}_{r+1}^{\pm}\mathbf{x}_s^{\pm} - q^{\pm 2}\mathbf{x}_s^{\pm}\mathbf{x}_{r+1}^{\pm} = q^{\pm 2}\mathbf{x}_r^{\pm}\mathbf{x}_{s+1}^{\pm} - \mathbf{x}_{s+1}^{\pm}\mathbf{x}_r^{\pm}, \\
[\mathbf{x}_r^{+}, \mathbf{x}_s^{-}] = \frac{\psi_{r+s}^{+} - \psi_{r+s}^{-}}{q - q^{-1}},
\end{gathered}
$$

where the $\psi_r^{\pm}$ are determined by equating powers of $u$ in the formal power series

$$\sum_{r=0}^{\infty} \psi_{\pm r}^{\pm} u^{\pm r} = K^{\pm 1}\exp\left(\pm(q - q^{-1})\sum_{s=1}^{\infty}\mathbf{h}_{\pm s}u^{\pm s}\right).$$

Define the $q$-divided powers

$$(\mathbf{x}_k^{\pm})^{(r)} = \frac{(\mathbf{x}_k^{\pm})^r}{[r]!},$$

for all $k \in \mathbf{Z}$, $r \geq 0$.

Define

$$\mathbf{\Lambda}^{\pm}(u) = \sum_{m=0}^{\infty}\mathbf{\Lambda}_{\pm m}u^m = \exp\left(-\sum_{k=1}^{\infty}\frac{\mathbf{h}_{\pm k}}{[k]}u^k\right).$$

The subalgebras $\mathbf{U}_q$, $\mathbf{U}_q^{\mathrm{fin}}$, $\mathbf{U}_q(<)$, $\mathbf{U}(0)$ etc., are defined in the obvious way. Let $\mathbf{U}_q^e(0)$ be the subalgebra of $\mathbf{U}_q^e$ generated by $\mathbf{U}(0)$, $K^{\pm 1}$ and $D^{\pm 1}$. The following result is a simple corollary of the PBW theorem for $\mathbf{U}_q^e$, [B].

**Lemma 1.1.** $\mathbf{U}_q^e = \mathbf{U}_q(<)\mathbf{U}_q^e(0)\mathbf{U}_q(>)$. □

For any invertible element $x \in \mathbf{U}_q^e$ and any $r \in \mathbf{Z}$, define

$$\begin{bmatrix} x \\ r \end{bmatrix} = \frac{xq^r - x^{-1}q^{-r}}{q - q^{-1}}.$$

Let $\mathbf{U}_{\mathbf{A}}^e$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}_q^e$ generated by the $K^{\pm 1}$, $(\mathbf{x}_k^{\pm})^{(r)}$ ($k \in \mathbf{Z}$, $r \geq 0$), $D^{\pm 1}$ and $\begin{bmatrix} D \\ r \end{bmatrix}$ ($r \in \mathbf{Z}$). Then, [L1], [BCP],

$$\mathbf{U}_q^e \cong \mathbf{U}_{\mathbf{A}}^e \otimes_{\mathbf{A}} \mathbf{K}.$$

Define $\mathbf{U}_{\mathbf{A}}(<)$, $\mathbf{U}_{\mathbf{A}}(0)$ and $\mathbf{U}_{\mathbf{A}}(>)$ in the obvious way. Let $\mathbf{U}_{\mathbf{A}}^e(0)$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}_{\mathbf{A}}$ generated by $\mathbf{U}_{\mathbf{A}}(0)$ and the elements $K^{\pm 1}$, $D^{\pm 1}$, $\begin{bmatrix} K \\ r \end{bmatrix}$ and $\begin{bmatrix} D \\ r \end{bmatrix}$ ($r \in \mathbf{Z}$). The following is proved as in Proposition 2.7 in [BCP].

**Proposition 1.1.** $\mathbf{U}_{\mathbf{A}}^e = \mathbf{U}_{\mathbf{A}}(<)\mathbf{U}_{\mathbf{A}}(0)\mathbf{U}_{\mathbf{A}}^e(\mathfrak{h})\mathbf{U}_{\mathbf{A}}(>)$. □

The next lemma is easily checked.

**Lemma 1.2.**

(i) *There is a unique* $\mathbf{C}$*-linear anti-automorphism* $\Psi$ *of* $\mathbf{U}_q^e$ *such that* $\Psi(q) = q^{-1}$ *and*

$$\Psi(K) = K, \quad \Psi(D) = D,$$
$$\Psi(x_r^{\pm}) = x_r^{\pm}, \quad \Psi(h_r) = -h_r,$$

*for all* $r \in \mathbf{Z}$.

(ii) *There is a unique* $\mathbf{K}$*-algebra automorphism* $\Phi$ *of* $\mathbf{U}_q^e$ *such that*

$$\Phi(\mathbf{x}_r^{\pm}) = \mathbf{x}_{-r}^{\pm}, \quad \Phi(\Lambda^{\pm}(u)) = \Lambda^{\mp}(u).$$

(iii) *For* $0 \neq a \in \mathbf{K}$*, there exists a* $\mathbf{K}$*-algebra automorphism* $\tau_a$ *of* $\mathbf{U}_q^e$ *such that*

$$\tau_a(\mathbf{x}_r^{\pm}) = a^r \mathbf{x}_r^{\pm}, \quad \tau_a(\mathbf{h}_r) = a^r \mathbf{h}_r, \quad \tau_a(K) = K, \quad \tau_a(D) = D,$$

*for* $r \in \mathbf{Z}$*. Moreover,*

$$\tau_a(\mathbf{\Lambda}_r) = a^r \mathbf{\Lambda}_r.$$

## 2. The modules $W_q(m)$

In this section, we recall the definition and elementary properties of the modules $W_q(\lambda)$ from [CP4], and state the main theorem of this paper.

**Definition 2.1.** A $\mathbf{U}_q^e$-module $V_q$ is said to be of *type* 1 if

$$V_q = \bigoplus_{\lambda \in P^e} (V_q)_\lambda,$$

where the weight space

$$(V_q)_\lambda = \{v \in V_q : K.v = q^{\lambda(h)} v, \quad D.v = q^{\lambda(d)} v\}.$$

A $\mathbf{U}_q^e$-module of type 1 is said to be *integrable* if the elements $\mathbf{x}_k^{\pm}$ act locally nilpotently on $V_q$ for all $k \in \mathbf{Z}$. The analogous definitions for $\mathbf{U}^e$, $\mathbf{U}^{\rm fin}$ and $\mathbf{U}_q^{\rm fin}$ are clear.

We shall only be interested in modules of type 1 in this paper. It is well known [L1] that, if $m \geq 0$, there is a unique irreducible $\mathbf{U}_q^{\rm fin}$-module $V_q^{\rm fin}(m)$, of dimension $m+1$, generated by a vector $v$ such that

$$K.v = q^m v, \quad x_0^+.v = 0, \quad (x_0^-)^{m+1}.v = 0.$$

Recall [L2] that, if $V_q$ is any integrable $sl_2$-module, then

$$\dim_{\mathbf{K}}(V_q)_n = \dim_{\mathbf{K}}(V_q)_{-n},$$

for all $n \in \mathbf{Z}$. Let $V(m)$ denote the $(m+1)$-dimensional irreducible representation of $sl_2$.

Define the following generating series in an indeterminate $u$ with coefficients in $\mathbf{U}_q$:

$$\tilde{\mathbf{X}}^-(u) = \sum_{m=-\infty}^{\infty} \mathbf{x}_m^- u^{m+1}, \qquad \mathbf{X}^-(u) = \sum_{m=1}^{\infty} \mathbf{x}_m^- u^m,$$
$$\mathbf{X}^+(u) = \sum_{m=0}^{\infty} \mathbf{x}_m^+ u^m, \qquad \mathbf{X}_0^-(u) = \sum_{m=0}^{\infty} \mathbf{x}_m^- u^{m+1},$$
$$\tilde{\mathbf{H}}(u) = \sum_{m=-\infty}^{\infty} \mathbf{h}_m u^{m+1}, \qquad \mathbf{\Lambda}^\pm(u) = \sum_{m=0}^{\infty} \mathbf{\Lambda}_{\pm m} u^m = \exp\left(-\sum_{k=1}^{\infty} \frac{\mathbf{h}_{\pm k}}{[k]} u^k\right).$$

Given a power series $f$ in $u$, we let $f_s$ denote the coefficient of $u^s$ in $f$.

For any integer $m \geq 0$, let $I_q^e(m)$ be the left ideal in $\mathbf{U}_q^e$ generated by the elements

$$\mathbf{x}_k^+ \quad (k \in \mathbf{Z}), \quad K - q^m, \quad D - 1,$$
$$\mathbf{\Lambda}_r \ (|r| > m), \quad \mathbf{\Lambda}_m \mathbf{\Lambda}_{-r} - \mathbf{\Lambda}_{m-r} \ (1 \leq r \leq m),$$
$$\left(\tilde{\mathbf{X}}_i^-(u)\mathbf{\Lambda}^+(u)\right)_r \mathbf{U}(0) \quad (r \in \mathbf{Z}), \quad \left(\mathbf{X}_0^-(u)^r \mathbf{\Lambda}^+(u)\right)_s \mathbf{U}(0) \quad (r \geq 1, \ |s| > m).$$

The ideal $I_q(m)$ in $\mathbf{U}_q$ is defined in the obvious way (by omitting $D$ from the definition).

Set

$$W_q(m) = \mathbf{U}_q^e / I_q^e(m) \cong \mathbf{U}_q / I_q(m).$$

Clearly, $W_q(m)$ is a left $\mathbf{U}_q^e$-module and a right $\mathbf{U}_q(0)$-module. Further, the left and right actions of $\mathbf{U}_q(0)$ on $W_q(m)$ commute. Let $w_m$ denote the image of 1 in $W_q(m)$. If $I_q(m,0)$ (resp. $I_{\mathbf{A}}(m,0)$) is the left ideal in $\mathbf{U}_q(0)$ (resp. $\mathbf{U}_{\mathbf{A}}(0)$) generated by the elements $\mathbf{\Lambda}_m$ $(|m| > \lambda(h))$ and $\mathbf{\Lambda}_{\lambda(h)}\mathbf{\Lambda}_{-m} - \mathbf{\Lambda}_{\lambda(h)-m}$ $(1 \leq m \leq \lambda(h))$, then

$$\mathbf{U}_q(0).w_m \cong \mathbf{U}_q(0)/I_q(m,0) \quad (\text{resp. } \mathbf{U}_{\mathbf{A}}(0).w_m \cong \mathbf{U}_{\mathbf{A}}(0)/I_{\mathbf{A}}(m,0))$$

as $\mathbf{U}_q(0)$-modules (resp. as $\mathbf{U}_{\mathbf{A}}(0)$-modules). The $\mathbf{U}^e$-modules $W(m)$ are defined in the analogous way.

Let $\mathbf{U}_q(+)$ be the subalgebra of $\mathbf{U}_q$ generated by the $\mathbf{x}_k\pm$ for $k \geq 0$. The subalgebras $\mathbf{U}_{\mathbf{A}}(+)$ and $\mathbf{U}(+)$ of $\mathbf{U}_{\mathbf{A}}$ and $\mathbf{U}$, respectively, are defined in the obvious way. The following proposition was proved in [CP4].

**Proposition 2.1.** *Let $m \geq 1$.*

(i) *We have*

$$\mathbf{U}_q(0)/I_q(m,0) \cong \mathbf{K}[\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \cdots, \mathbf{\Lambda}_m, \mathbf{\Lambda}_m^{-1}],$$
$$\mathbf{U}_{\mathbf{A}}(m,0)/I_{\mathbf{A}}(m,0) \cong \mathbf{A}[\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \cdots, \mathbf{\Lambda}_m, \mathbf{\Lambda}_m^{-1}],$$

*as algebras over $\mathbf{K}$ and $\mathbf{A}$, respectively.*

(ii) *The $\mathbf{U}^e$-module $W_q(m)$ is integrable for all $m \geq 0$.*

(iii) *$W_q(m) = \mathbf{U}_q(+).w_m$. In fact, $W_q(m)$ is spannned over $\mathbf{K}$ by the elements*

$$(x_0^-)^{(r_0)}(x_1^-)^{(r_1)} \cdots (x_{m-1}^-)^{(r_{m-1})}\mathbf{U}_q(0).w_m,$$

*where $r_j \geq 0$, $\sum_j r_j \leq m$.*

*Analogous results hold for the $\mathbf{U}$-modules $W(m)$.* $\square$

Let $\mathcal{P}_m$ be the Laurent polynomial ring in $m$ variables with complex coefficients. The symmetric group $\Sigma_m$ acts on it in the obvious way; let $\mathcal{P}_m^{\Sigma_m}$ be the ring of symmetric Laurent polynomials. In view of Proposition 2.1, we see that

$$\mathbf{U}_q(0)/I_q(m,0) \cong \mathbf{K}\mathcal{P}_m^{\Sigma_m}, \quad \mathbf{U}_{\mathbf{A}}/I_{\mathbf{A}}(m,0) \cong \mathbf{A}\mathcal{P}_m^{\Sigma_m},$$

where $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$ denotes $\mathcal{P}_m^{\Sigma_m} \otimes \mathbf{K}$, etc.

Let $V$ be the two-dimensional irreducible $sl_2$-module with basis $v_0,\ v_1$ such that

$$\begin{gathered} x^+.v_0 = 0, \quad h.v_0 = v_0, \quad x^-.v_0 = v_1, \\ x^+.v_1 = v_0, \quad h.v_1 = -v_1, \quad x^-.v_1 = 0. \end{gathered}$$

Let $L(V) = V \otimes \mathbf{C}[t,t^{-1}]$ be the $L(sl_2)$-module defined in the obvious way. Let $T^m(L(V))$ be the $m$-fold tensor power of $L(V)$ and let $S^m(L(V))$ be its symmetric part. Then, $T^m(L(V))$ is a left $\mathbf{U}$-module and a right $\mathcal{P}_m$-module, and $S^m(L(V))$ is a left $\mathbf{U}$-module and a right $\mathcal{P}_m^{\Sigma_m}$-module. The following was proved in [CP4].

**Theorem 1.** As left $\mathbf{U}$-modules and right $\mathcal{P}_m^{\Sigma_m}$-modules, we have

$$W(m) \cong S^m(L(V)).$$

In particular, $W(m)$ is a free $\mathcal{P}_m^{\Sigma_m}$-module of rank $2^m$. $\square$

Our goal in this paper is to prove an analogue of this result for the $W_q(m)$. To do this, we introduce a suitable quantum analogue of $S^m(L(V))$ by using the Hecke algebra and a certain quantum symmetrizer.

**Definition 2.2.** The *Hecke algebra* $\mathcal{H}_m$ is the associative unital algebra over $\mathbf{C}(q)$ generated by elements $T_i$ $(i = 1, 2, \dots, m-1)$ with the following defining relations:

$$\begin{aligned} (T_i + 1)(T_i - q^2) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j &= T_j T_i \quad \text{if } |i-j| > 1. \end{aligned}$$

Set $L_q(V) = L(V) \otimes \mathbf{K}$. It is easily checked that the following formulas define an action of $\mathbf{U}_q^e$ on $L_q(V)$:

$$x_k^{\pm}.(v_\pm \otimes t^r) = 0, \ \ x_k^{\pm}.(v_\mp \otimes t^r) = v_\pm \otimes t^{k+r}, \tag{2.1}$$

$$\Psi^+(u).(v_\pm \otimes t^r) = v_\pm \otimes \frac{q^{\pm1} - q^{\mp1} tu}{1 - tu} t^r, \tag{2.2}$$

$$\Psi^-(u).(v_\pm \otimes t^r) = v_\pm \otimes \frac{q^{\mp1} - q^{\pm1} t^{-1} u}{1 - t^{-1} u} t^r. \tag{2.3}$$

The $m$-fold tensor product $T^m(L_q(V))$ is a left $\mathbf{U}_q^e$-module (the action being given by the comultiplication of $\mathbf{U}_q$) and a right $\mathcal{P}_m$-module (in the obvious way). Now, as a vector space over $\mathbf{K}$,

$$L_q(V)^{\otimes m} \cong V^{\otimes m} \otimes_{\mathbf{K}} \mathbf{K}[t_1^{\pm1}, \dots, t_m^{\pm1}],$$

and $\Sigma_m$ acts naturally (on the right) on both $V^{\otimes m}$ and $\mathbf{K}[t_1^{\pm1}, \dots, t_m^{\pm1}]$ by permuting the variables. If $\mathbf{v} \in V^{\otimes m}$ and $f \in \mathbf{K}[t_1^{\pm1}, \dots, t_m^{\pm1}]$, denote the action of $\sigma \in \Sigma_m$ by $\mathbf{v}^\sigma$ and $f^\sigma$, respectively. Let $\sigma_i$ be the transposition $(i, i+1) \in \Sigma_m$.

**Proposition 2.2.** *([KMS, Section 1.2]) The Hecke algebra $\mathcal{H}_m$ acts on $L_q(V)^{\otimes m}$ on the right, the action of the generators being given as follows :*

$$(v_{t_1}\otimes\cdots\otimes v_{t_m}\otimes f).T_i=\begin{cases}-q(v_{t_1}\otimes\cdots\otimes v_{t_m})^{\sigma_i}\otimes f^{\sigma_i}\\ \qquad -(q^2-1)(v_{t_1}\otimes\cdots\otimes v_{t_m})\otimes\frac{t_{i+1}f^{\sigma_i}-t_if}{t_i-t_{i+1}}\\ \qquad\qquad \text{if } t_i=+,\ t_{i+1}=-,\\ -v_{t_1}\otimes\cdots\otimes v_{t_m}\otimes f^{\sigma_i}\\ \qquad -(q^2-1)(v_{t_1}\otimes\cdots\otimes v_{t_m})\otimes\frac{t_i(f^{\sigma_i}-f)}{t_i-t_{i+1}}\\ \qquad\qquad \text{if } t_i=t_{i+1},\\ -q(v_{t_1}\otimes\cdots\otimes v_{t_m})^{\sigma_i}\otimes f^{\sigma_i}\\ \qquad -(q^2-1)(v_{t_1}\otimes\cdots\otimes v_{t_m})\otimes\frac{t_i(f^{\sigma_i}-f)}{t_i-t_{i+1}}\\ \qquad\qquad \text{if } t_i=-,\ t_{i+1}=+.\end{cases}$$

*Moreover, this action commutes with the left action of $\mathbf{U}_q^e$ on $L_q(V)$ and the right action of $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$.* □

As is well known, the second and third relations in the definition of $\mathcal{H}_m$ imply that, if $\sigma=\sigma_{i_1}\dots\sigma_{i_N}$ is a reduced expression for $\sigma\in\Sigma_m$, so that $N$ is the length $\ell(\sigma)$, the element $T_\sigma=T_{i_1}\dots T_{i_N}\in\mathcal{H}_m$ depends only on $\sigma$, and is independent of the choice of its reduced expression. We define the symmetrizing operator

$$\mathcal{S}^{(m)}:L_q(V)^{\otimes m}\to L_q(V)^{\otimes m}$$

by

$$\mathcal{S}^{(m)}=\frac{1}{[m]!}\sum_{\sigma\in\Sigma_m}(-q^{-2})^{\ell(\sigma)}T_\sigma.$$

**Proposition 2.3.** *As left $\mathbf{U}_q^e$-modules and right $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-modules, we have*

$$L_q(V)^{\otimes m}=im(\mathcal{S}^{(m)})\oplus ker(\mathcal{S}^{(m)}).$$

*Proof.* It is clear from Proposition 2.2 that $\mathrm{im}(\mathcal{S}^{(m)})$ and $\ker(\mathcal{S}^{(m)})$ are submodules for both the right and left actions.

The following proof is adapted from that of Proposition 1.1 in [KMS]. For each $i=1,\dots,m-1$, we have a factorization

$$\mathcal{S}^{(m)}=\left(\sum_{\sigma'}(-q^{-2})^{\ell(\sigma')}T_{\sigma'}\right)(1-q^{-2}T_i),$$

where $\sigma'$ ranges over $\Sigma_m/\{1,\sigma_i\}$. From this and the first of the defining relations of $\mathcal{H}_m$, it follows that

$$\mathcal{S}^{(m)}(T_i+1)=0.$$

In other words, $T_i$ acts on the right on $\mathrm{im}(\mathcal{S}^{(m)})$ as multiplication by $-1$. It follows that $\mathcal{S}^{(m)}$ acts on $\mathrm{im}(\mathcal{S}^{(m)})$ by multiplication by the scalar

$$\frac{1}{[m]!}\sum_{\sigma\in\Sigma_m}(q^{-2})^{\ell(\sigma)}=\frac{1}{[m]!}\prod_{l=1}^{m}\frac{1-q^{-2l}}{1-q^{-2}}=q^{-m(m-1)/2}.$$

Hence,

$$\mathcal{S}^{(m)}(\mathcal{S}^{(m)}-q^{-m(m-1)/2})=0,$$

and this implies the proposition. $\square$

As in [KMS], define an ordered basis $\{u_m\}_{m\in\mathbf{Z}}$ of $L_q(V)$ by setting

$$u_{-2r} = v_+ \otimes t^r, \quad u_{1-2r} = v_- \otimes t^r.$$

Let $u_{r_1} \otimes_S \cdots \otimes_S u_{r_m}$ be the image of $u_{r_1} \otimes \cdots \otimes u_{r_m}$ under the projection of $L_q(V)^{\otimes m}$ onto $L_q(V)^{\otimes m}/\ker(\mathcal{S}^{(m)})$. By Proposition 2.3, this can be identified with an element, which we also denote by $u_{r_1} \otimes_S \cdots \otimes_S u_{r_m}$, in $\mathrm{im}(\mathcal{S}^{(m)})$.

**Proposition 2.4.** *The set $\{u_{r_1} \otimes_S \cdots \otimes_S u_{r_m} : r_1 \geq \cdots \geq r_m\}$ is a basis of the vector space im($\mathcal{S}^{(m)}$). Further, im($\mathcal{S}^{(m)}$) is a free $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-module on $2^m$ generators.*

*Proof.* The first statement in proved as in [KMS], Proposition 1.3. As for the second, for any $0 \leq s \leq m$, let $\mathrm{im}(\mathcal{S}^{(m)})_s$ be the subspace spanned by $u_{r_1} \otimes_S \cdots \otimes_S u_{r_m}$, where exactly $s$ of the $r_i$ are even. This space is naturally isomorphic as a right $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-module to $\mathbf{K}\mathcal{P}_m^{\Sigma_s\times\Sigma_{m-s}}$. But this module is well-known to be free of rank $\binom{m}{s}$. $\square$

Let $\mathbf{w} = u_0 \otimes_S \cdots \otimes_S u_0$. Then, $\mathbf{w}$ satisfies the defining relations of $W_q(m)$, so we have a map of left $\mathbf{U}_q^e$-modules and right $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-modules $\eta_m : W_q(m) \to \mathrm{im}(\mathcal{S}^{(m)})$ that takes $w_m$ to $\mathbf{w}$. The main theorem of this paper is

**Theorem 2.** The map $\eta_m$ is an isomorphism. In particular, $W_q(m)$ is a free $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-module of rank $2^m$.

The theorem is deduced from the following two lemmas.

**Lemma 2.1.** *Let $\mathfrak{m}$ be any maximal ideal in $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$, and let $d$ be the degree of the field extension $\mathbf{K}\mathcal{P}_m^{\Sigma_m}/\mathfrak{m}$ of $\mathbf{K}$. Then,*

$$dim_{\mathbf{K}} \frac{W_q(m)}{W_q(m)\mathfrak{m}} = 2^m d.$$

**Lemma 2.2.** *The map $\eta_m$ is surjective.*

We defer the proofs of these lemmas to the next section. Once we have these two lemmas, the proof of Theorem 2 is completed in exactly the same way as Theorem 1. We include it here for completeness.

*Proof of Theorem 2.* Let $K$ be the kernel of $\eta_m$. Since $\mathrm{im}(\mathcal{S}^{(m)})$ is a free, hence projective, right $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-module by Proposition 2.4, it follows that

$$W_q(m) = \mathrm{im}(\mathcal{S}^{(m)}) \oplus K,$$

as right $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$-modules. Let $\mathfrak{m}$ be any maximal ideal in $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$. It follows from Lemma 2.1 and Proposition 2.4 that

$$K/K\mathfrak{m} = 0$$

as vector spaces over $\mathbf{K}$. Since this holds for all maximal ideals $\mathfrak{m}$, Nakayama's lemma implies that $K = 0$, proving the theorem. $\square$

## 3. Proof of lemmas 2.1 and 2.2

In preparation for the proof of Lemma 2.1, we first show that the modules in question are finite-dimensional. Recall that a maximal ideal in $\mathbf{K}\mathcal{P}_m^{\Sigma_m}$ is defined by an $m$-tuple of points $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_m)$, with $\pi_m \neq 0$, in an algebraic closure $\overline{\mathbf{K}}$ of $\mathbf{K}$, i.e., it is the kernel of the homomorphism $ev_{\boldsymbol{\pi}} : \mathbf{K}\mathcal{P}_m^{\Sigma_m} \to \overline{\mathbf{K}}$ that sends $\boldsymbol{\Lambda}_i \to \pi_i$. Let $\mathbf{F}_{\boldsymbol{\pi}}$ be the smallest subfield of $\overline{\mathbf{K}}$ containing $\mathbf{K}$ and $\pi_1, \cdots, \pi_m$. Clearly, $\mathbf{F}_{\boldsymbol{\pi}}$ is a finite-rank $\mathbf{U}_q(0)$-module. Set

$$W_q(\boldsymbol{\pi}) = W_q(m) \otimes_{\mathbf{U}_q(0)} \mathbf{F}_{\boldsymbol{\pi}},$$

and let $w_{\boldsymbol{\pi}} = w_m \otimes 1$. The $\mathbf{U}$-modules $W(\pi)$ are defined similarly (with $\pi \in \mathbf{C}^m$).

The following lemma is immediate from Proposition 2.1.

**Lemma 3.1.** *We have*

$$\mathbf{U}_q(0).w_{\boldsymbol{\pi}} = \mathbf{F}_{\boldsymbol{\pi}} w_{\boldsymbol{\pi}}.$$

*Further,* $W_q(\boldsymbol{\pi})$ *is spanned over* $\mathbf{F}_{\boldsymbol{\pi}}$ *by the elements*

$$(x_0^-)^{(r_0)}(x_1^-)^{(r-1)} \cdots (x_{m-1}^-)^{(r_{m-1})}$$

*with* $\sum_i r_i \leq m$.

*In particular,* $dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) < \infty$. ∎

The modules $W_q(m)$ and $W_q(\boldsymbol{\pi})$, together with their classical analogues, have the following universal properties.

**Proposition 3.1.** *Let* $\lambda \in P_e^+$.

- (i) *Let* $V_q$ *be any integrable* $\mathbf{U}_q^e$*-module generated by an element* $v$ *of* $(V_q)_m$ *satisfying* $\mathbf{U}_q(>).v = 0$. *Then,* $V_q$ *is a quotient of* $W_q(m)$.
- (ii) *Let* $V_q$ *be a finite-dimensional quotient* $\mathbf{U}_q$*-module of* $W_q(m)$ *and let* $v$ *be the image of* $w_m$ *in* $V_q$. *Assume that* $ker(ev_{\boldsymbol{\pi}}).v = 0$ *for some* $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_m)$, *where the* $\pi_i \in \overline{\mathbf{K}}$. *Then,* $V_q$ *is a quotient of* $W_q(\boldsymbol{\pi})$.
- (iii) *Let* $V_q$ *be finite-dimensional* $\mathbf{U}_q$*-module generated by an element* $v \in (V_q)_m$ *and such that* $\mathbf{U}_q(>).v = 0$ *and* $ker(ev_{\boldsymbol{\pi}}).v = 0$ *for some* $\boldsymbol{\pi}$. *Then,* $V_q$ *is a quotient of* $W_q(\boldsymbol{\pi})$.

*Analogous statements hold in the classical case.*

*Proof.* This proposition was proved in [CP4] in the case when $\boldsymbol{\pi} \in \mathbf{K}^m$. The proof in this case is identical, and follows immediately from the defining relations of $W_q(m)$ and $W_q(\boldsymbol{\pi})$. ∎

One can now deduce the following theorem, which classifies the irreducible finite-dimensional representations of $\mathbf{U}_q$ over $\mathbf{K}$.

**Theorem 3.** Let $\boldsymbol{\pi} \in \overline{\mathbf{K}}^m$ be as above. Then, $W_q(\boldsymbol{\pi})$ has a unique irreducible quotient $\mathbf{U}_q$-module $V_q(\boldsymbol{\pi})$. Conversely, any irreducible finite-dimensional $\mathbf{U}_q$-module is isomorphic to $V_q(\boldsymbol{\pi})$ for a suitable choice of $\boldsymbol{\pi}$.

*Proof.* To prove that $W_q(\boldsymbol{\pi})$ has a unique irreducible quotient, it suffices to prove that it has a unique maximal $\mathbf{U}_q$-submodule. For this, it suffices to prove that, if $N$ is any submodule, then

$$N \cap W_q(\boldsymbol{\pi})_m = \{0\}.$$

Since $W_q(\boldsymbol{\pi})_m = \mathbf{U}_q(0).w_{\boldsymbol{\pi}}$ is an irreducible $\mathbf{U}_q(0)$-module, it follows that

$$N \cap W_q(\boldsymbol{\pi})_m \neq \{0\} \implies w_{\boldsymbol{\pi}} \in N,$$

and hence that $N = W_q(\boldsymbol{\pi})$. Conversely, if $V$ is any finite-dimensional irreducible module, one can show as in [CP1], [CP4] that there exists $0 \neq v \in V_m$ such that $\mathbf{U}_q(>).v = 0$ and that $\boldsymbol{\Lambda}_r.v = 0$ if $|r| > m$. This shows that $V_m$ must be an irreducible module for $\mathbf{K}[\boldsymbol{\Lambda}_1, \cdots, \boldsymbol{\Lambda}_m, \boldsymbol{\Lambda}_m^{-1}]$, and the result follows. $\square$

It follows from the preceding discussion that, to prove Lemma 2.1, we must show that, if $\mathbf{F}_{\boldsymbol{\pi}}$ is an extension of $\mathbf{K}$ of degree $d$, then

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) = 2^m d. \tag{3.1}$$

Assume from now on that we have a fixed finite extension $\mathbf{F}$ of $\mathbf{K}$ of degree $d$ and an element $\boldsymbol{\pi} \in \mathbf{F}^m$ as above. Given $0 \neq a \in \mathbf{K}$, and $\boldsymbol{\pi} \in \mathbf{F}^m$ where $\mathbf{K} \subset \mathbf{F}$, define

$$\boldsymbol{\pi}_a = (a\boldsymbol{\pi}_1, a^2\boldsymbol{\pi}_2, \cdots, a^m\boldsymbol{\pi}_m).$$

Given any $\mathbf{U}_q$-module $M$, and $0 \neq a \in \mathbf{K}$, let $\tau_a^* M$ be the $\mathbf{U}_q$-module obtained by pulling back $M$ through the automorphism $\tau_a$ defined in Lemma 1.2. The next lemma is immediate from Proposition 3.1.

**Lemma 3.2.** *We have*

$$\tau_a^* W_q(m) \cong W_q(m), \quad \tau_a^* W_q(\boldsymbol{\pi}) \cong W_q(\boldsymbol{\pi}_a),$$

*where the first isomorphism is one of $\mathbf{U}_q^e$-modules and the second is an isomorphism of $\mathbf{U}_q$-modules.* $\square$

Let $\overline{\mathbf{A}}$ be the integral closure of $\mathbf{A}$ in $\mathbf{F}$. Fix $a \in \mathbf{A}$ such that $\boldsymbol{\pi}_a \in \overline{\mathbf{A}}^m$. By Lemma 3.2, to prove (3.1) it suffices to prove that

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}_a) = 2^m d.$$

Let $\mathbf{L} \supset \mathbf{K}$ be the smallest subfield of $\mathbf{F}$ such that $\boldsymbol{\pi}_a \in \mathbf{L}^m$ and let $\tilde{\mathbf{A}}$ be the integral closure of $\mathbf{A}$ in $\mathbf{L}$. Then, $\tilde{\mathbf{A}}$ is free of rank $d$ as an $\mathbf{A}$-module and

$$\mathbf{L} \cong \tilde{\mathbf{A}} \otimes_{\mathbf{A}} \mathbf{K}.$$

In what follows we write $\boldsymbol{\pi}$ for $\boldsymbol{\pi}_a$. Set

$$W_{\mathbf{A}}(\boldsymbol{\pi}) = \mathbf{U}_{\mathbf{A}} \otimes_{\mathbf{U}_{\mathbf{A}}(0)} \tilde{\mathbf{A}} w_{\boldsymbol{\pi}}.$$

By Lemma 3.1, $W_{\mathbf{A}}(\boldsymbol{\pi})$ is finitely-generated as an $\tilde{\mathbf{A}}$-module, and hence as an $\mathbf{A}$-module. Further,

$$W_q(\boldsymbol{\pi}) \cong W_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbf{K}$$

as vector spaces over $\mathbf{K}$. Note, however, that $W_{\mathbf{A}}(\boldsymbol{\pi})$ is not an $\mathbf{U}_{\mathbf{A}}$-module in general, since $\pi_m^{-1}$ need not be in $\tilde{\mathbf{A}}$. However, $W_{\mathbf{A}}(\boldsymbol{\pi})$ is a $\mathbf{U}_{\mathbf{A}}(+)$-module and

$$W_q(\boldsymbol{\pi}) \cong W_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbf{K},$$

as $\mathbf{U}_q(+)$-modules.

Set

$$\mathbf{U}_1(+) = \mathbf{U}_{\mathbf{A}}(+) \otimes_{\mathbf{A}} \mathbf{C}_1.$$

This is essentially the universal enveloping algebra $\mathbf{U}(+)$ of $sl_2 \otimes \mathbf{C}[t]$, and hence

$$\overline{W_q(\boldsymbol{\pi})} = W_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbf{C}_1$$

is a module for $\mathbf{U}(+)$.

Since

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) = \mathrm{rank}_{\mathbf{A}} W_{\mathbf{A}}(\boldsymbol{\pi}) = \dim_{\mathbf{C}} \overline{W_q(\boldsymbol{\pi})},$$

it suffices to prove that

$$\dim_{\mathbf{C}} \overline{W_q(\boldsymbol{\pi})} = 2^m d.$$

Define elements $\Lambda_r \in \mathbf{U}(+)$ in the same way as the elments $\boldsymbol{\Lambda}_r$ are defined, replacing $q$ by 1.

**Lemma 3.3.** *With the above notation, there exists a filtration*

$$\overline{W_q(\boldsymbol{\pi})} = W_1 \supset W_2 \supset \cdots \supset W_d \supset W_{d+1} = 0$$

*such that, for each $i = 1, \ldots, d$, $W_i/W_{i+1}$ is generated by a non-zero vector $v_i$ such that*

$$x_r^+.v_i = 0, \quad (x_r^-)^{m+1}.v_i = 0 \quad (r \geq 0), \tag{3.2}$$

$$h_0.v_i = mv_i, \quad \Lambda_r.v_i = \lambda_{i,r} v_i \quad (r > 0), \tag{3.3}$$

*where the $\lambda_{i,r} \in \mathbf{C}$ and $\lambda_{i,r} = 0$ for $r > m$.*

*Proof.* Let $\overline{W_q(\boldsymbol{\pi})}_n$ be the eigenspace of $h_0$ acting on $\overline{W_q(\boldsymbol{\pi})}$ with eigenvalue $n \in \mathbf{Z}$. Of course,

$$\overline{W_q(\boldsymbol{\pi})} = \bigoplus_{n=-m}^{m} \overline{W_q(\boldsymbol{\pi})}_n.$$

We can choose a basis $w_1, w_2, \ldots, w_l$, say, of $\overline{W_q(\boldsymbol{\pi})}_m$ such that the action of $\Lambda_i$, for $i = 1, \ldots, m$, is in upper triangular form. Let $W_i$ be the $\mathbf{U}(+)$-submodule of $\overline{W_q(\mathfrak{m})}$ generated by $\{w_i, w_{i+1}, \ldots, w_l\}$. This gives a filtration with the stated properties. To see that $l = d$, note that $W_{\mathbf{A}}(\boldsymbol{\pi})_m = \tilde{\mathbf{A}} w_m$ is a free $\mathbf{A}$-module of rank $d$, hence

$$\overline{W_q(\boldsymbol{\pi})}_m = W_{\mathbf{A}}(\boldsymbol{\pi})_m \otimes_{\mathbf{A}} \mathbf{C}_1$$

is a vector space of dimension $d$. □

**Lemma 3.4.** *Let $\pi = 1 + \sum_{r=1}^{n} \lambda_r u^r \in \mathbf{C}[u]$ be a polynomial of degree $n$, and let $m \geq n$. Let $W_+(\pi, m)$ be the quotient of $\mathbf{U}(+)$ by the left ideal generated by the elements*

$$h - m, \quad \Lambda_r - \lambda_r, \quad x_r^+, \quad (x_r^-)^{m+1},$$

*for all $r \geq 0$. Then,*

$$dim_{\mathbf{C}} W_+(\pi, m) \leq 2^m.$$

*Proof.* This is exactly the same as the proof given in [CP5, Sections 3 and 6] that $\dim_{\mathbf{C}} W(\pi) \leq 2^{\deg(\pi)}$. We note that the arguments used there only make use of elements of the subalgebra $\mathbf{U}(+)$ of $\mathbf{U}$. □

It follows immediately from this lemma that

$$\dim_{\mathbf{C}} \overline{W_q(\boldsymbol{\pi})} \leq 2^m d.$$

Indeed, each $W_i/W_{i+1}$ in Lemma 3.3 is clearly a quotient of some $W_+(\pi, m)$ satisfying the conditions of Lemma 3.4, and so has dimension $\leq 2^m$.

We have now proved that

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) \leq 2^m d.$$

To prove the reverse inequality, let $\tilde{\mathbf{F}}$ be the splitting field of the polynomial $1 + \sum_{i=1}^m \pi_i u^i$ over $\mathbf{F}$, say

$$1 + \sum_{i=1}^m \pi_i u^i = \prod_{i=1}^m (1 - a_i u),$$

with $a_1, \ldots, a_m \in \tilde{\mathbf{F}}$. Let $V_{\mathbf{F}}(a_i)$ be a two-dimensional vector space over $\tilde{\mathbf{F}}$ with basis $\{v_+, v_-\}$, define an action of $\mathbf{U}_q$ on it by setting $t = a_i$ in the formulas in (2.1), (2.2) and (2.3), and set

$$\tilde{W} = \bigotimes_{i=1}^m V_{\tilde{\mathbf{F}}}(a_i).$$

Clearly,

$$\dim_{\mathbf{K}} \tilde{W} = 2^m d\tilde{d},$$

where $\tilde{d}$ is the degree of $\tilde{\mathbf{F}}$ over $\mathbf{F}$. If $\{f_1, \ldots, f_{\tilde{d}}\}$ is a basis of $\tilde{\mathbf{F}}$ over $\mathbf{F}$, and if $\tilde{w} = v_+^{\otimes m}$, then

$$\tilde{W} = \bigoplus_{j=1}^{\tilde{d}} \tilde{W}_j,$$

where $\tilde{W}_j$ is the $\mathbf{U}_q$-submodule of $\tilde{W}$ generated by $f_j \tilde{w}$ (see [CP3, Proof of 2.5]). Moreover, the vectors $f_j \tilde{w}$ satisfy the defining relations of $W_q(\boldsymbol{\pi})$, and so are quotients of $W_q(\boldsymbol{\pi})$. It follows that

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) \geq 2^m d.$$

The proof of Lemma 2.1 is now complete. □

Turning to Lemma 2.2, set

$$L_{\mathbf{A}}(V) = V \otimes \mathbf{A}[t, t^{-1}].$$

Clearly, $L_{\mathbf{A}}(V)$ is a $\mathbf{U}_{\mathbf{A}}$-module. The map $\mathcal{S}^{(m)}$ takes $L_{\mathbf{A}}(V)^{\otimes m}$ into itself; set

$$\operatorname{im}(\mathcal{S}^{(m)}) = S_q(m), \quad S_{\mathbf{A}}(m) = \mathcal{S}_q(m) \cap L_{\mathbf{A}}(V)^{\otimes m}.$$

We have

$$S_{\mathbf{A}}^{(m)} \otimes_{\mathbf{A}} \mathbf{K} \cong S_q^{(m)}, \quad S_{\mathbf{A}}^{(m)} \otimes_{\mathbf{A}} \mathbf{C}_1 \cong S^m(L(V)). \tag{3.4}$$

The first isomorphism above is clear; the second requires the basis constructed in Proposition 2.4. The proof of Proposition 2.3 shows that

$$L_{\mathbf{A}}(V)^{\otimes m} = \mathcal{S}_{\mathbf{A}}(m) \oplus (\ker(\mathcal{S}_q^{(m)}) \cap L_{\mathbf{A}}(V)^{\otimes m}).$$

Given $\boldsymbol{\pi} \in \mathbf{F}^m$ such that $\pi_i \in \overline{\mathbf{A}}$, set

$$S_q(\boldsymbol{\pi}) = S_q(m) \otimes_{\mathbf{U}_q(0)} \mathbf{F}, \quad S_{\mathbf{A}}(\boldsymbol{\pi}) = S_{\mathbf{A}} \otimes_{\mathbf{U}_{\mathbf{A}}(0)} \tilde{\mathbf{A}}.$$

Then, $S_q(\boldsymbol{\pi})$ (resp. $S_{\mathbf{A}}(\boldsymbol{\pi})$) is a $\mathbf{U}_q$-module (resp. $\mathbf{U}_{\mathbf{A}}(+)$-module) and

$$S_q(\boldsymbol{\pi}) \cong S_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbf{K} \tag{3.5}$$

as $\mathbf{U}_q(+)$-modules. Further, the map $\eta_m : W_q(m) \to S_q(m)$ induces a map $\eta_{\boldsymbol{\pi}} : W_q(\boldsymbol{\pi}) \to S_q(\boldsymbol{\pi})$ that takes $W_{\mathbf{A}}(\boldsymbol{\pi})$ into $S_{\mathbf{A}}(\boldsymbol{\pi})$.

Set $\overline{\mathbf{F}} = \mathbf{F} \otimes_{\mathbf{A}} \mathbf{C}_1$. Let $\overline{\boldsymbol{\pi}} : \mathbf{C}[\Lambda_1, \cdots, \Lambda_m] \to \overline{\mathbf{F}}$ be the homomorphism obtained by sending $\Lambda_i$ to $\pi_i \otimes 1$ and set

$$S(\overline{\boldsymbol{\pi}}) = S^m(L(V)) \otimes_{\mathbf{U}(0)} \overline{\mathbf{F}}.$$

Now, in [CP4] we proved that $S^m(L(V))$ is a free $\mathbf{C}[\Lambda_1, \cdots, \Lambda_m]$-module of rank $2^m$, hence $S(\overline{\boldsymbol{\pi}})$ has dimension $2^m d$. Further, [CP4],

$$W(\overline{\boldsymbol{\pi}}) \cong S(\overline{\boldsymbol{\pi}}) = \mathbf{U}(+).v_+^{\otimes m}.$$

This shows that the induced map $\overline{\eta_{\boldsymbol{\pi}}} : \overline{W_q(\boldsymbol{\pi})} \to \overline{S_q(\boldsymbol{\pi})}$ is surjective and hence, using Lemma 2.1, that it is an isomorphism.

Let $K_q(\boldsymbol{\pi})$ be the kernel of $\eta_{\boldsymbol{\pi}}$ and let $K_{\mathbf{A}}(\boldsymbol{\pi}) = K_q(\boldsymbol{\pi}) \cap W_q(\boldsymbol{\pi})$. Then, $K_{\mathbf{A}}(\boldsymbol{\pi})$ is free $\mathbf{A}$-module and

$$\dim_{\mathbf{K}} K_q(\boldsymbol{\pi}) = \mathrm{rank}_{\mathbf{A}} K_{\mathbf{A}}(\boldsymbol{\pi}).$$

The previous argument shows that

$$\overline{K_q(\boldsymbol{\pi})} = K_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbf{C}_1$$

is zero. Hence, $K_q(\boldsymbol{\pi}) = 0$ and the map $\eta_{\boldsymbol{\pi}}$ is an isomorphism for all $\boldsymbol{\pi} \in \overline{A}^m$. But now, by twisting with an automorphism $\tau_a$ for $0 \neq a \in \mathbf{K}$, we have a commutative diagram

$$\begin{array}{ccc} W_q(\boldsymbol{\pi}_a) & \longrightarrow & S_q(\boldsymbol{\pi}_a) \\ \downarrow & & \downarrow \\ W_q(\boldsymbol{\pi}) & \longrightarrow & S_q(\boldsymbol{\pi}) \end{array}$$

for *any* $\boldsymbol{\pi} \in \mathbf{F}^m$, in which the vertical maps are isomorphisms of $\mathbf{U}_q(+)$-modules. If $a$ is such that $\boldsymbol{\pi}_a \in \overline{\mathbf{A}}^m$, the top horizontal map is also an isomorphism, hence so is the bottom horizontal map. Thus, $W_q(\boldsymbol{\pi}) \to S_q(\boldsymbol{\pi})$ is an isomorphism for *all* $\boldsymbol{\pi} \in \mathbf{F}^m$. It follows from Nakayama's lemma that $\eta_m : W_q(m) \to S_q^{(m)}$ is surjective and the proof of Lemma 2.2 is complete. ☐

## 4. The general case: a conjecture

In this section, we indicate to what extent the results of this paper can be generalized to the higher rank cases, and then state a conjecture in the general case.

Thus, let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank $n$ of type $A$, $D$ or $E$ and let $\hat{\mathfrak{g}}$ be the corresponding untwisted affine Lie algebra. Given any dominant integral weight $\lambda$ for $\mathfrak{g}$, one can define an integrable $\mathbf{U}_q(\hat{\mathfrak{g}})$-module $W_q(\lambda)$ on which the centre acts trivially, [CP4]. These modules have a family of finite-dimensional quotients $W_q(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = (\pi^1, \cdots, \pi^n)$ and the $\pi^i \in \overline{\mathbf{K}}^{\lambda(i)}$. The module $W_q(\boldsymbol{\pi})$ has a unique irreducible quotient $V_q(\boldsymbol{\pi})$ and one can prove the analogue of Theorem 3. (The proofs of these statements are the same as in the $sl_2$ case.)

We make the following

**Conjecture.** For any $\boldsymbol{\pi}$ as above,

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) = m_\lambda,$$

where $m_\lambda \in \mathbf{N}$ is given by

$$m_\lambda = \prod_{i=1}^{n} (m_i)^{\lambda_i}, \quad m_i = \dim_{\mathbf{K}} W_q(i),$$

and $W_q(i)$ is the finite-dimensional module associated to the $n$-tuple $(\pi^1, \cdots, \pi^n)$ with $\pi^j = \{0\}$ if $j \neq i$ and $\pi^i = \{1\}$. $\square$

In the case of $sl_2$, the conjecture is established in this paper. It follows from the results in [C] that $W_q(i)$ is in fact an irreducible $\mathbf{U}_q(\hat{\mathfrak{g}})$-module and hence [CP2] the values of the $m_i$ are actually known. The results of [C] also establish the conjecture for all $\boldsymbol{\pi}$ associated to the fundamental weight $\lambda_i$ of $\mathfrak{g}$, for all $i = 1, \cdots, n$.

Using the results in [VV], one can show that

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) \geq m_\lambda.$$

It suffices to prove the reverse inequality in the case when the $\pi^i \in \overline{\mathbf{A}}^{\lambda(i)}$ for all $i$. One can prove exactly as in this paper that the $\mathbf{U}_q(+)$-modules $W_q(\boldsymbol{\pi})$ admit an $\mathbf{U}_{\mathbf{A}}(+)$-lattice $W_{\mathbf{A}}(\boldsymbol{\pi})$, so that

$$\dim_{\mathbf{K}} W_q(\boldsymbol{\pi}) = \operatorname{rank}_{\mathbf{A}} W_{\mathbf{A}}(\boldsymbol{\pi}) = \dim_{\mathbf{C}} \overline{W_q(\boldsymbol{\pi})}.$$

Thus, it suffices to prove the conjecture in the classical case, i.e.,

$$\dim_{\mathbf{C}} W(\boldsymbol{\pi}) = m_\lambda,$$

where $m_\lambda$ is defined above.

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